1. Determine if the following are linear mappings:

a) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 5x_1 \end{pmatrix}$.

b) $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 + x_3 \end{pmatrix}$.

c) $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

d) $f: \mathbb{P}_2 \to \mathbb{P}_1$, $f(a_0 + a_1 t + a_2 t^2) = 2a_0 + a_1 + (a_2 - a_1)t$.

e) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ x_2 \end{pmatrix}$.

f) $f: M_{n,n} \longrightarrow M_{n,n}$, $f(A) = A + A^t$

$M_{n,n}$: vector space of square matrices $n \times n$

2. Consider the subset M of 2×2 square matrices defined as:

$$M = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \text{ con } a,b,c \in \mathbb{R} \right\}$$

and the mapping $M \to M$ defined by:

$$f\left(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \right) = \begin{pmatrix} a-b & b-a \\ c & 0 \end{pmatrix}.$$

Observe that M is a vector space and consider the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Show that f is linear and compute its associated matrix with respect to B.

3. Consider the linear mapping $f: \mathbb{R}^3 \to \mathbb{R}^2$ that maps the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

to the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

respectively. These vectors are coordinate vectors with respect to the canonical bases of $\mathbb{R}^3$ y $\mathbb{R}^2$. Compute the associated matrix of f with respect to the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

4. Define the linear mapping $f: V \to W$, where $\dim V = 3$ and $\dim W = 4$, such that $f(\bar{b}_1 - \bar{b}_3) = \bar{c}_1$, $f(\bar{b}_2 - \bar{b}_3) = \bar{c}_1 - \bar{c}_2$, and $f(2\bar{b}_3) = 2\bar{c}_1 + 2\bar{c}_3$. Here, $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ is a basis of V and $C = \{\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4\}$ is a basis of W.

a) Find the associated matrix with respect to $\mathcal{B}$ and C.

b) Find a basis for $\operatorname{Im} f$

c) Find a basis for $\operatorname{Ker} f$

5. Determine if the following mappings:

a) $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 + x_2 \\ -x_3 \end{pmatrix}$,

b) $f: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}$,

are linear mappings. If they are linear, find their kernel and range. Study their injectivity and surjectivity.

6. Determine whether the following statements are true or false. Justify your answer.

a) If $g: V \to W$ is a linear mapping, sometimes it is possible to find three distinct vectors $\bar{u}, \bar{v} \in V$ and $\bar{w} \in W$ such that $g(\bar{u}) = g(\bar{v}) = \bar{w}$.

b) Assuming that the previous statement is true, if $g(\bar{u}) = g(\bar{v}) = \bar{w}$, then $\bar{u} - \bar{v} \in \ker g$.

c) If $g: V \to W$ is a linear mapping, then the range of g is W.

d) If $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ is a basis of $\mathbb{R}^n$ and $\{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n\}$ is a basis of $\mathbb{P}_{n-1}$, then there are two linear mappings $f: \mathbb{R}^n \to \mathbb{P}_{n-1}$ and $g: \mathbb{P}_{n-1} \to \mathbb{R}^n$ such that $f(\bar{v}_i) = \bar{w}_i$ and $g(\bar{w}_i) = \bar{v}_i$ for $i = 1, 2, \dots, n$.

e) If $f:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping defined by $f(\bar{0})=\bar{0}$, then f is identically the null mapping ($f(\bar{x})=\bar{0}$ $\forall \bar{x} \in \mathbb{R}^2$).

f) There is a linear mapping $f:\mathbb{R}^5 \to \mathbb{R}^5$ with $\dim \ker f = \dim \operatorname{Im} f$.

g) Assuming that $f: M_{2,2} \to M_{2,2}$ is linear with $\dim \operatorname{Im} f = 4$, if $f(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

7. Define the linear mapping $f: V \to W$ where $\dim V = \dim W = 3$, such that $f(\bar{e}_1) = \bar{u}_1 - \bar{u}_2$, $f(\bar{e}_2) = \bar{u}_2$, and $f(\bar{e}_3) = \bar{u}_1$. Here, $\mathcal{B} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is a basis of V and $C = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is a basis of W.

a) Find the associated matrix with respect to $\mathcal{B}$ y C.

b) Find $\dim \operatorname{Im} f$.

c) Find a basis of $\operatorname{Ker} f$.

d) Given a new basis of V, $\tilde{\mathcal{B}} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ where $\bar{e}_1 = \bar{v}_1$, $\bar{e}_2 = \bar{v}_1 + \bar{v}_2$, and $\bar{e}_3 = \bar{v}_1 + \bar{v}_3$, compute the associated matrix with respect to $\tilde{\mathcal{B}}$ and C.

e) Given the following change of basis

$$[\bar{w}]_{\tilde{C}} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\bar{w}]_C, \quad \bar{w} \in W$$

compute the associated matrix with respect to $\mathcal{B}$ and $\tilde{\mathcal{C}}$.

g) Obtain the matrix associated with f with respect to $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$.

h) study the injectivity and surjectivity of f.

8. In $\mathbb{R}^3$, consider the basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$. Study the injectivity and surjectivity of the linear mapping $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\bar{e}_1) = a\bar{e}_1 + \bar{e}_2 + \bar{e}_3$, $f(\bar{e}_2) = \bar{e}_1 + \bar{e}_2 + \bar{e}_3$, and $f(\bar{e}_3) = \bar{e}_1 + b\bar{e}_2 + \bar{e}_3$ in terms of the parameters a and b.